Finitely Generated Abelian Groups

$(G,+)$ – Abelian group

Additive Notation : Given $\in G$, $n \in \mathbb{Z}$

$$nx = \begin{cases} x + x \cdots + x \ (n \text{ times}) & \text{if } n > 0 \\ 0 \leftarrow e & \text{if } n = 0 \\ (-x) + \cdots + (-x) \ (-n \text{ times}) & \text{if } n < 0 \end{cases}$$

inverse of x

Carefull, not multiplication

usually written xH

$H \subset G$ subgroup, $x \in G$, $\quad x + H = \{x + h \mid h \in G\}$

Always normal

$x + H = y + H \iff x - y \in H$

Aim : Classify all finitely generated Abelian groups up to isomorphism.

We say x is a torsion element of G

Definition $\quad tG := \{ x \in G \mid \mathrm{ord}(x) < \infty \} \subset G$

Proposition $\quad tG \subset G$ is a subgroup, called the torsion subgroup.

Proof

$x \in tG \iff \exists\, n \in \mathbb{N}$ such that $nx = 0$

- $\mathrm{ord}(0) = 1 \Rightarrow 0 \in tG$
- $x, y \in tG \Rightarrow \exists\, m, n \in \mathbb{N}$ such that $mx = 0 = ny$

  $\Rightarrow (mn)(x+y) \overset{\text{Abelian}}{=} n(mx) + m(ny) = 0 + 0 = 0$

  $\Rightarrow x + y \in tG$
- $x \in tG \Rightarrow \exists\, m \in \mathbb{N}$ such that $mx = 0 \Rightarrow m(-x) = 0$

  $\Rightarrow -x \in tG$ $\quad\square$

Example

$G = (\mathbb{R}/\mathbb{Z}, +)$. $\quad [x] \in tG \iff \exists\, n \in \mathbb{N}$ such that

$n[x] = [nx] = [0] \iff \exists\, n \in \mathbb{N}$ such that $nx \in \mathbb{Z} \iff x \in \mathbb{Q}$

$\Rightarrow tG = (\mathbb{Q}/\mathbb{Z}, +)$

Remarks

- If G non-Abelian, tG may not be a subgroup.

Definition

G is torsion $\iff G = tG$

G is torsion-free $\iff tG = \{0\}$

Examples

- $|G| < \infty \Rightarrow G$ torsion
- $(\mathbb{Q}/\mathbb{Z}, +)$ torsion
- $(\mathbb{Z}^n, +)$ torsion-free

Proposition G/tG torsion-free.

Proof

Let $x + tG$ be torsion in G/tG

$\Rightarrow \exists\, n \in \mathbb{N}$ such that $nx + tG = 0 + (tG)$

$\Rightarrow nx \in tG$

$\Rightarrow \exists\, m \in \mathbb{N}$ such that $m(nx) = 0 \Rightarrow (mn)x = 0$

$\Rightarrow x \in tG \Rightarrow x + (tG) = 0 + (tG)$

$\square$

Definition

(f.g.)

A finitely generated, Abelian group G is **free** if $\exists\, x_1, \ldots, x_n \in G$ such that the following property holds:

- Given $g \in G$ $\exists!\, \lambda_i \in \mathbb{Z}$ such that

$$g = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n$$

We call $\{x_1, \ldots, x_n\}$ a $\mathbb{Z}$-basis for G.

Remark

- $\{x_1, \ldots, x_n\} \subset G$ a $\mathbb{Z}$-basis $\Rightarrow \operatorname{ord}(x_i) = \infty \;\forall i$

  and $G = gp(x_1) \oplus \cdots \oplus gp(x_n) \cong (\mathbb{Z}^n, +)$

Proposition Let G be a finitely generated free Abelian group.
Any two $\mathbb{Z}$-bases have the same size.

Proof

Let $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_m\}$ be $\mathbb{Z}$-bases for G

Define $2G := \{2g \mid g \in G\}$ ← normal subgroup of G

$2G = \{\lambda_1 x_1 + \dots + \lambda_n x_n \mid 2 \mid \lambda_i\}$

Given $a, b \in G$, $a = \alpha_1 x_1 + \dots + \alpha_n x_n$, $b = \beta_1 x_1 + \dots + \beta_n x_n$

$$a + 2G = b + 2G \iff a - b = (\alpha_1 - \beta_1)x_1 + \dots + (\alpha_n - \beta_n)x_n \in 2G$$
$$\iff 2 \mid \alpha_i - \beta_i \quad \forall i$$

$$\Rightarrow \left| G/2G \right| = 2^n$$

Exactly the same logic applied to $\{y_1, \dots, y_m\}$ implies

$$\left| G/2G \right| = 2^m \Rightarrow 2^n = 2^m \Rightarrow n = m \qquad \square$$

Definition finitely generated free Abelian

$\text{rank}(G)$ = size of any $\mathbb{Z}$-basis ← Analogue of dimension in linear algebra

Theorem

G finitely generated free Abelian $\iff$ G finitely generated and torsion free

Proof See Notes

Theorem G finitely generated Abelian $\Rightarrow$ G/tG finitely generated free Abelian

Proof

$\text{gp}(\{x_1, \dots, x_n\}) = G \Rightarrow \text{gp}(\{x_1 + tG, \dots, x_n + tG\}) = G/tG$

Hence G finitely generated $\Rightarrow$ G/tG finitely generated

G/tG torsion-free $\Rightarrow$ G/tG free Abelian. $\square$

Definition finitely generated Abelian

$$\text{rank}(G) := \text{rank}(G/tG)$$

Theorem Let G be f.g. Abelian with $\operatorname{rank}(G) = n$. Then $\exists\ F \subset G$ a finitely generated free Abelian subgroup such that

$$G = F \oplus tG \quad \text{and} \quad \operatorname{rank}(F) = n$$

Proof

G/tG finitely generated, free Abeian $\Rightarrow \exists\ x_1, \dots, x_n \in G$ such that $\{x_1 + tG, \dots, x_n + tG\}$ is a $\mathbb{Z}$-basis for G/tG.

Let $F = gp(\{x_1, \dots, x_n\})$.

Claim $\{x_1, \dots, x_n\} \subset F$ is a $\mathbb{Z}$-basis for F.

$\{x_1(tG), \dots, x_n(tG)\}$ a $\mathbb{Z}$-basis for G/tG

$$\lambda_1 x_1 + \dots + \lambda_n x_n = \lambda_1' x_n + \dots + \lambda_n' x_n$$

in $\mathbb{Z}$

$$\Rightarrow \lambda_1(x_1 + tG) + \dots + \lambda_n(x_n + tG) = \lambda_1'(x_1 + tG) + \dots + \lambda_n'(x_n + tG)$$

$\Rightarrow \lambda_i = \lambda_i' \quad \forall i \quad \Rightarrow \{x_1, \dots, x_n\}$ is a $\mathbb{Z}$-basis for F.

We must now prove $G = F \oplus tG$. Note G Abelian $\Rightarrow$ 2/ holds

We should check 1/

- Let $g \in G \Rightarrow \exists\ \lambda_1, \dots, \lambda_n \in \mathbb{Z}$ such that

$$\begin{aligned} g + tG &= \lambda_1(x_1 + tG) + \dots + \lambda_n(x_n + tG) \\ &= (\lambda_1 x_1 + \dots + \lambda_n x_n) + tG \end{aligned}$$

$$\Rightarrow g = \underbrace{\lambda_1 x_1 + \dots + \lambda_n x_n}_{\in F} + \underbrace{h}_{\in tG}, \text{ where } h \in tG.$$

$$\underset{\substack{\uparrow\\ F}}{f_1} + \underset{\substack{\uparrow\\ tG}}{h_1} = \underset{\substack{\uparrow\\ F}}{f_2} + \underset{\substack{\uparrow\\ tG}}{h_2} \Rightarrow \underset{\substack{\uparrow\\ F}}{f_1 - f_2} = \underset{\substack{\uparrow\\ tG}}{h_2 - h_1}$$

F is torsion free

$$\Rightarrow \operatorname{ord}(f_1 - f_2) < \infty \Rightarrow f_1 - f_2 = 0 \Rightarrow f_1 = f_2$$

$$\Rightarrow h_1 = h_2$$

$$\Rightarrow G = F \oplus tG \qquad \square$$

<u>Corollary</u> Let G be a finitely generated Abelian group then $G \cong \mathbb{Z}^n \times tG$ where $n = \text{rank}(G)$ and $|tG| < \infty$

<u>Proof</u> $G = F \oplus tG$, hence we can define the homomorphism

$$\begin{aligned} \phi : G &\longrightarrow tG \\ \underset{F}{f} + \underset{tG}{h} &\longmapsto h \end{aligned}$$

$\text{Im}\,\phi = tG$, $\ker\phi = F$ $\underset{\text{1st Iso Theorem}}{\Rightarrow}$ $G/F \cong tG$

G f.g. $\Rightarrow$ G/F f.g. $\Rightarrow$ tG f.g.

$gp(x_1, \dots, x_n) = tG$
$\Rightarrow tG = \{\lambda_1 x_1 + \dots + \lambda_n x_n \mid 0 \leq \lambda_i \leq \text{ord}(x_i)\}$

tG f.g. and torsion $\Rightarrow$ $|tG| < \infty$

$\text{rank}(G) = n \Rightarrow F \cong \mathbb{Z}^n$

$G = F \oplus tG \cong \underbrace{\mathbb{Z}^n}_{\text{f.g. Free Abelian}} \times \underbrace{tG}_{\text{finite Abelian}}$ $\square$

Hence to classify f.g. Abelian groups we must now classify all finite Abelian groups.